

AUTOMORPHISMS OF THE PLANE PRESERVING A CURVE

JÉRÉMY BLANC AND IMMANUEL STAMPFLI

ABSTRACT. We study the group of automorphisms of the affine plane preserving some given curve, over any field. The group is proven to be algebraic, except in the case where the curve is a bunch of parallel lines. Moreover, a classification of the groups of positive dimension occurring is also given in the case where the curve is geometrically irreducible and the field is perfect.

1. INTRODUCTION

Let \mathbf{k} be any field. This article studies (closed) curves $\Gamma \subset \mathbb{A}^2 = \mathbb{A}_{\mathbf{k}}^2$ and the group of automorphisms of \mathbb{A}^2 (defined over \mathbf{k}) which preserve this curve. We will denote this group by $\text{Aut}(\mathbb{A}^2, \Gamma)$. In other words, we study polynomials in $\mathbf{k}[x, y]$ and the \mathbf{k} -algebra automorphisms of $\mathbf{k}[x, y]$ that send the polynomial on a multiple of itself. We will always assume that the curve is reduced, i.e. that the polynomial does not contain any multiple factor. For our purpose, it is a natural assumption.

If Γ has equation in $\mathbf{k}[x]$, we will say that Γ is a *fence*. In this case, $\text{Aut}(\mathbb{A}^2, \Gamma)$ is easy to describe, and it is in fact an infinite union of algebraic groups. Our main result consists of showing that this is the only case where such phenomenon occurs.

Recall that $\text{Aut}(\mathbb{A}^2)$ has the structure of an ind-variety. More precisely, the set $\text{Aut}(\mathbb{A}^2)_d$ of automorphisms of degree $\leq d$ is an algebraic variety and $\text{Aut}(\mathbb{A}^2)_d$ is closed in $\text{Aut}(\mathbb{A}^2)_{d+1}$ for any d . This gives to $\text{Aut}(\mathbb{A}^2) = \bigcup_{d=1}^{\infty} \text{Aut}(\mathbb{A}^2)_d$ the structure of an ind-variety, and since composition and taking inverse preserve this structure, $\text{Aut}(\mathbb{A}^2)$ is an ind-group (see [Kum02, Chapter IV] for precise definitions of “ind-variety” and “ind-group”).

By definition, an algebraic subgroup of $\text{Aut}(\mathbb{A}^2)$ is a closed subgroup of bounded degree. Let us recall that the group $\text{Aut}(\mathbb{A}^2)$ contains the following natural algebraic subgroups

$$\begin{aligned} \text{Aff}(\mathbb{A}^2) &= \{(x, y) \mapsto (ax + by + e, cx + dy + f) \mid a, b, c, d, e, f \in \mathbf{k}, ad - bc \neq 0\}, \\ J_n &= \{(x, y) \mapsto (ax + P(y), by + c) \mid a, b \in \mathbf{k}^*, c \in \mathbf{k}, P \in \mathbf{k}[y], \deg(P) \leq n\}. \end{aligned}$$

Moreover, $\text{Aut}(\mathbb{A}^2)$ is generated by the union of these groups (Jung - van der Kulk’s Theorem [Jun42], [vdK53]), any algebraic subgroup of $\text{Aut}(\mathbb{A}^2)$ is contained in one of these subgroups [Kam79, Theorem 4.3].

For any curve $\Gamma \subset \mathbb{A}^2$, the group $\text{Aut}(\mathbb{A}^2, \Gamma)$ is a closed ind-subgroup of $\text{Aut}(\mathbb{A}^2)$. The question is to know if this is an algebraic subgroup, i.e. if it has bounded degree.

Theorem 1. *Let Γ be a curve in \mathbb{A}^2 . Applying an automorphism of \mathbb{A}^2 , one of the following holds:*

Date: April 10, 2013.

2010 Mathematics Subject Classification. 14R10, 14R20, 14H37, 14H50, 14J50, 14E07.

The authors gratefully acknowledge support by the Swiss National Science Foundation Grants “Birational Geometry” 128422 and “Automorphisms of Affine n-Space” 137679.

- i) The curve Γ has equation $F(x) = 0$, where $F(x) \in \mathbf{k}[x]$ and $\text{Aut}(\mathbb{A}^2, \Gamma) = \{(x, y) \mapsto (ax+b, cy+P(x)) \mid a, c \in \mathbf{k}^*, b \in \mathbf{k}, P \in \mathbf{k}[x], F(ax+b)/F(x) \in \mathbf{k}^*\}$.
- ii) The group $\text{Aut}(\mathbb{A}^2, \Gamma)$ is equal to $\{g \in \text{Aff}(\mathbb{A}^2) \mid g(\Gamma) = \Gamma\}$ or $\{g \in J_n \mid g(\Gamma) = \Gamma\}$ for some integer n . Moreover, the action of $\text{Aut}(\mathbb{A}^2, \Gamma)$ on Γ gives an isomorphism of $\text{Aut}(\mathbb{A}^2, \Gamma)$ with a closed subgroup of $\text{Aut}(\Gamma)$, (this latter being an algebraic group).

In particular, $\text{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group if and only if there is no automorphism of \mathbb{A}^2 which sends Γ onto a fence.

Remark that the existence of an automorphism of \mathbb{A}^2 which sends Γ onto a line implies that $\Gamma \simeq \mathbb{A}^1$, and that the converse is true in characteristic 0 by the Abhyankar-Moh Theorem [AM75], but false in general. So Theorem 1 implies, in positive characteristic, that non-trivial embeddings of \mathbb{A}^1 into \mathbb{A}^2 are rigid in the sense that they admit only few compatible automorphisms of \mathbb{A}^2 .

Another consequence of Theorem 1 is that the fixed locus of an automorphism of \mathbb{A}^2 only contains points and curves equivalent to lines, a result already observed by Friedland and Milnor in [FM89], in the case where $\mathbf{k} = \mathbb{C}$ (see [Jel03] for some generalisations to higher dimensions).

Remark that in the case where $\mathbf{k} = \mathbb{C}$, the observation on fixed points of Friedland and Milnor, the Abhyankar-Moh Theorem and the Lin-Zaidenberg Theorem [ZL83] imply Theorem 1 in the case where the curve Γ has an irreducible component non-isomorphic to \mathbb{C}^* . The interesting case of Theorem 1 is then, if $\mathbf{k} = \mathbb{C}$, the description of $\text{Aut}(\mathbb{A}^2, \Gamma)$ when Γ is \mathbb{C}^* (or a union of such curves). Note that there exist only partial classifications of the closed embeddings of \mathbb{C}^* into \mathbb{A}^2 (see [CNKR09]), and that there exist complicated torsion-abelian closed subgroups of $\text{Aut}(\mathbb{A}^2)$ which are not conjugated to a subgroup of $\text{Aff}(\mathbb{A}^2)$ or to the Jonquières group $\bigcup_{n=1}^{\infty} J_n$ (see [Wri79]). Theorem 1 implies that such groups do not preserve any curve.

The proof of Theorem 1 is done in Section 3, using tools of birational geometry of surfaces introduced in Section 2.

In Section 4, we refine the theorem by describing more precisely the possibilities for the group $\text{Aut}(\mathbb{A}^2, \Gamma)$, in the case where Γ is geometrically irreducible and the ground field \mathbf{k} is perfect. We obtain then the following result (the tori T of cases iv), v) are precisely described in Proposition 4.5).

Theorem 2. *Let Γ be a geometrically irreducible closed curve in \mathbb{A}^2 , defined over the perfect field \mathbf{k} . Applying an automorphism of \mathbb{A}^2 , one of the following holds:*

- i) *The curve Γ is the line with equation $x = 0$ and $\text{Aut}(\mathbb{A}^2, \Gamma) = \{(x, y) \mapsto (ax, by + P(x)) \mid a, b \in \mathbf{k}^*, P \in \mathbf{k}[x]\}$.*
- ii) *The curve Γ has equation $x^b = \lambda y^a$, where $\lambda \in \mathbf{k}^*$ and $a, b > 1$ are coprime integers. Moreover, $\text{Aut}(\mathbb{A}^2, \Gamma)$ is equal to the group \mathbf{k}^* acting diagonally via $(x, y) \mapsto (t^a x, t^b y)$.*
- iii) *The curve Γ has equation $x^b y^a = \lambda$, where $\lambda \in \mathbf{k}^*$ and $a, b \geq 1$ are coprime integers. Moreover, $\text{Aut}(\mathbb{A}^2, \Gamma)$ contains the group \mathbf{k}^* acting diagonally via $(x, y) \mapsto (t^a x, t^{-b} y)$, and is equal to this group if $(a, b) \neq (1, 1)$, or is the group $\mathbf{k}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ generated by \mathbf{k}^* and by $(x, y) \mapsto (y, x)$ if $(a, b) = (1, 1)$.*
- iv) *The curve Γ has equation $\lambda x^2 + \nu y^2 = 1$, where $\lambda, \nu \in \mathbf{k}^*$, $-\lambda\nu$ is not a square in \mathbf{k} and $\text{char}(\mathbf{k}) \neq 2$. The group $\text{Aut}(\mathbb{A}^2, \Gamma)$ is the subgroup of $\text{GL}(2, \mathbf{k})$*

preserving the form $\lambda x^2 + \nu y^2$, which is isomorphic to $T \rtimes \mathbb{Z}/2\mathbb{Z}$ for some non- \mathbf{k} -split torus T .

- v) The curve Γ has equation $x^2 + \mu xy + y^2 = 1$, where $\mu \in \mathbf{k}^*$ and $x^2 + \mu x + 1$ has no root in \mathbf{k} , and $\text{char}(\mathbf{k}) = 2$. The group $\text{Aut}(\mathbb{A}^2, \Gamma)$ is the subgroup of $\text{GL}(2, \mathbf{k})$ preserving the form $x^2 + \mu xy + y^2$, which is isomorphic to $T \rtimes \mathbb{Z}/2\mathbb{Z}$ for some non- \mathbf{k} -split torus T .
- vi) The group $\text{Aut}(\mathbb{A}^2, \Gamma)$ is a zero-dimensional (hence finite) subgroup of $\text{Aff}(\mathbb{A}^2)$ or J_n for some n .

2. REMINDERS ON COMPLETIONS OF \mathbb{A}^2

In this section, we define natural completions of \mathbb{A}^2 , and links between them.

2.1. Natural completions of \mathbb{A}^2 .

Example 2.1. The morphism

$$\begin{aligned} \mathbb{A}^2 &\hookrightarrow \mathbb{P}^2 \\ (x, y) &\mapsto (x : y : 1) \end{aligned}$$

yields an isomorphism $\mathbb{A}^2 \xrightarrow{\sim} \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$, where $L_{\mathbb{P}^2}$ is the line of \mathbb{P}^2 with equation $z = 0$.

Example 2.2. For $n \geq 1$, the n -th Hirzebruch surface \mathbb{F}_n is

$$\mathbb{F}_n = \{((a : b : c), (u : v)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid bv^n = cu^n\}.$$

Let $E_n, L_{\mathbb{F}_n} \subset \mathbb{F}_n$ be the curves given by $(1 : 0 : 0) \times \mathbb{P}^1$ and $v = 0$, respectively. The morphism

$$\begin{aligned} \mathbb{A}^2 &\hookrightarrow \mathbb{F}_n \\ (x, y) &\mapsto ((x : y^n : 1), (y : 1)) \end{aligned}$$

gives an isomorphism $\mathbb{A}^2 \xrightarrow{\sim} \mathbb{F}_n \setminus (E_n \cup L_{\mathbb{F}_n})$. Note that E_n is the unique section of $\pi : \mathbb{F}_n \rightarrow \mathbb{P}^1$ of self-intersection $-n$, and $L_{\mathbb{F}_n}$ is a smooth fibre, and has thus self-intersection 0.

Remark 2.3. In Example 2.2, we could also have chosen $n = 0$, which yields the surface \mathbb{F}_0 , isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, via $((x : y : z), (u : v)) \mapsto ((x : y), (u : v))$, but we will not need this one in the sequel.

Definition 2.4. A *natural completion* (of \mathbb{A}^2) is a pair (X, B) which is given in Example 2.1 or Example 2.2: either $(X, B) = (\mathbb{P}^2, L_{\mathbb{P}^2})$ or $(X, B) = (\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n})$ for some $n \geq 1$. The isomorphism $\mathbb{A}^2 \rightarrow X \setminus B$ given in the examples will be called *canonical isomorphism*, or *canonical embedding of \mathbb{A}^2 into X* .

A birational map (respectively birational morphism) $(X, B) \dashrightarrow (X', B')$ between two natural completions is a birational map (respectively birational morphism) $X \dashrightarrow X'$ inducing an isomorphism $X \setminus B \rightarrow X' \setminus B'$.

We denote by $\text{Aut}(X, B)$ the group of automorphisms of (X, B) : it is the group of automorphisms of X which leave B (or equivalently $X \setminus B = \mathbb{A}^2$) invariant.

Remark 2.5. Given two natural completions $(X, B), (X', B')$, any birational map $\varphi : (X, B) \dashrightarrow (X', B')$ restricts to an isomorphism $X \setminus B \rightarrow X' \setminus B'$, and corresponds thus, via the canonical isomorphisms $\mathbb{A}^2 \simeq X \setminus B$ and $\mathbb{A}^2 \simeq X' \setminus B'$, to a unique automorphism of \mathbb{A}^2 . Moreover, every automorphism of \mathbb{A}^2 is obtained in this way.

Let us recall some easy fact on the automorphisms of natural completions:

Lemma 2.6. *Let (X, B) be a natural completion of \mathbb{A}^2 , and let $\iota: \mathbb{A}^2 \rightarrow X \setminus B$ be the canonical isomorphism associated. The group $\text{Aut}(X, B)$ corresponds via ι to the following subgroups of $\text{Aut}(\mathbb{A}^2)$ (see the introduction for the definition of $\text{Aff}(\mathbb{A}^2)$ and J_n):*

- 1) $X = \mathbb{P}^2$, $\text{Aut}(X, B) = \text{Aff}(\mathbb{A}^2) \simeq \text{GL}(2, \mathbf{k}) \ltimes \mathbf{k}^2$
- 2) $X = \mathbb{F}_n$, $\text{Aut}(X, B) = J_n \simeq (\mathbf{k}^*)^2 \ltimes (\mathbf{k} \ltimes \mathbf{k}^{n+1})$.

Proof. The first assertion follows from the fact that $\text{Aut}(\mathbb{P}^2)$ is the group of linear automorphisms, equal to $\text{PGL}(3, \mathbf{k})$.

Recall that the \mathbb{P}^1 -bundle structure given by the projection on the second factor is unique, and thus that the group of automorphisms of \mathbb{F}_n preserve the \mathbb{P}^1 -bundle structure (since $n \geq 1$). In particular, all automorphisms of $\text{Aut}(\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n})$ restrict on \mathbb{A}^2 to automorphisms preserving the \mathbb{A}^1 -bundle $\mathbb{A}^2 \rightarrow \mathbb{A}^1, (x, y) \mapsto y$. The precise description on the degree follows from a straightforward calculation. \square

2.2. Elementary links. There are two kinds of very simple birational maps between natural completions. The first one are automorphisms, that we described in Lemma 2.6, and the second one are elementary links, that we describe now. The results of this section are classical, we just remind them to the reader for self-containedness and in order to fix notation.

Definition 2.7. A *link* is a birational map $\varphi: (X, B) \dashrightarrow (X', B')$ between two natural completions, which is not an isomorphism, such that both φ and φ^{-1} have at most one $\bar{\mathbf{k}}$ -base-point (and in particular there is no infinitely near base-point). An *elementary link* is a link which does not decompose into $\varphi = \varphi' \circ \varphi''$, where φ', φ'' are links.

One can easily see that Definition 2.7 is equivalent to the classical definitions of elementary links which appear in the minimal model program (see [Cor95, Theorem of page 225] and [Isk96, Definition 2.2, page 597]):

Lemma 2.8. *Any elementary link is one of the following three types:*

- i) *link of type I: a map $(\mathbb{P}^2, L_{\mathbb{P}^2}) \dashrightarrow (\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1})$ consisting of blowing-up one \mathbf{k} -rational point of $L_{\mathbb{P}^2}$.*
- ii) *link of type II: maps $(\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n}) \dashrightarrow (\mathbb{F}_m, E_m \cup L_{\mathbb{F}_m})$ given by the blow-up of a \mathbf{k} -rational point p of $L_{\mathbb{F}_n}$ followed by the contraction of the strict transform of $L_{\mathbb{F}_n}$. Moreover, $m = n + 1$ if $p \in E_n$ and $m = n - 1$ if $p \notin E_n$.*
- iii) *link of type III: a morphism $(\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1}) \rightarrow (\mathbb{P}^2, L_{\mathbb{P}^2})$ given by the contraction of the curve E_1 onto a \mathbf{k} -rational point of $L_{\mathbb{P}^2}$.*

The inverse of a link of type I, II, III is a link of type III, II, I respectively.

Proof. Let $\varphi: (X, B) \dashrightarrow (X', B')$ be a birational map, which is an elementary link.

Suppose first that φ is a morphism, which implies that φ is the contraction of a (-1) -curve onto a \mathbf{k} -rational point. This implies that $X = \mathbb{F}_1$ and $X' = \mathbb{P}^2$, and that we are in the third case. If φ^{-1} is a morphism, we get symmetrically the first case.

In the remaining cases, both φ and φ^{-1} have exactly one base-point. These points are thus defined over \mathbf{k} , and both maps contract one irreducible curve. The curves have thus either self-intersection -1 or self-intersection 0 , depending if they contain the base-points. In the first case, we get a link $(\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1}) \dashrightarrow (\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1})$, which is not an elementary link since it factors through \mathbb{P}^2 as the composition

of links of type III and I. In the second case, we get a link of type II described above. \square

Proposition 2.9. *Let $\varphi: (X, B) \dashrightarrow (X', B')$ be a birational map between two natural completions of \mathbb{A}^2 . If φ is not an isomorphism, then there exist $m \geq 1$ and elementary links $\varphi_1, \dots, \varphi_m$ such that $\varphi = \varphi_m \cdots \varphi_1$.*

Remark 2.10. We call $\varphi = \varphi_m \cdots \varphi_1$ a *decomposition into m elementary links*. If φ is an isomorphism, we sometimes say that it decomposes into 0 elementary links. This is coherent with the fact that the composition of an elementary link with an isomorphism of completions is an elementary link.

Proposition 2.9 implicitly follows from the work made in [Lam02] or from [BD11, Theorem 3.0.2]. The proof given here is however direct. Note also that Jung-van der Kulk's Theorem is a direct consequence of this Proposition.

Proof. As all birational maps between projective smooth surfaces, φ admits a minimal resolution, i.e. two birational morphisms $\epsilon: Y \rightarrow X$, $\eta: Y \rightarrow X'$ such that $\varphi = \eta\epsilon^{-1}$, and such that no (-1) -curve of Y (not necessarily defined over \mathbf{k}) is contracted by both ϵ and η . Moreover, each point blown-up by η and ϵ belongs to the boundaries B, B' , as proper or infinitely near points.

We proceed by induction on the number of $\overline{\mathbf{k}}$ -points blown-up by η and ϵ , corresponding to the number of base-points of φ^{-1} and φ . If η is an isomorphism, φ is only a sequence of blow-ups in the boundary of B . Because of the nature of B and B' , this implies that φ is either an isomorphism or a link of type I, from $X = \mathbb{P}^2$ to $X' = \mathbb{F}_1$. Similarly, φ is an isomorphism or a link of type III if ϵ is an isomorphism.

We can thus assume that η (respectively ϵ) contract at least one (-1) -curve of Y , not contracted by ϵ (respectively η), and which is thus the strict transform of an irreducible curve $E \subset B$ (respectively $E' \subset B'$) of self-intersection ≥ -1 . If $E^2 = -1$, we factorise Y with a link of type III from $X = \mathbb{F}_1$ to \mathbb{P}^2 . We do the same with φ^{-1} if $(E')^2 = -1$.

Looking at the self-intersections of the curves of the boundaries B and B' , the remaining case is when $E^2 \geq 0$, $(E')^2 \geq 0$ and only one (-1) -curve of Y is contracted by η (respectively ϵ). This implies that φ and φ^{-1} have exactly one proper base-point. If $E^2 = 1$, we factorise φ with a link of type I from $X = \mathbb{P}^2$ to \mathbb{F}_1 . Otherwise, $E^2 = 0$ and we factorise φ with a link of type II from $X = \mathbb{F}_n$ to \mathbb{F}_m with $m = n \pm 1$. It remains to see that $m = 0$ is impossible. Indeed, otherwise the (-1) -curve of X would not pass through the unique proper base-point of φ and would thus be sent by φ onto a curve of self-intersection ≥ 0 , not contracted by φ^{-1} . \square

Corollary 2.11. *Let $\varphi: (X, B) \dashrightarrow (X', B')$ be a birational map between two natural completions of \mathbb{A}^2 . All $\overline{\mathbf{k}}$ -base-points of φ (that belong to X as proper or infinitely near points) are defined over \mathbf{k} .*

Proof. Follows from Proposition 2.9 and from the fact that the base-points of any elementary link are defined over \mathbf{k} (Lemma 2.8). \square

For our purpose, we will need the decomposition into elementary links given by Proposition 2.9, but we will also need to have more precise information on the composition of links and their base-points, provided by Lemma 2.12 and Proposition 2.14 below.

Lemma 2.12. *Let $\varphi: (X_0, B_0) \dashrightarrow (X_1, B_1)$ and $\psi: (X_1, B_1) \dashrightarrow (X_2, B_2)$ be two elementary links. The birational map $\psi \circ \varphi$ is an isomorphism if and only if one of the following occurs:*

- i) $X_1 = \mathbb{F}_1$, $X_0 = X_2 = \mathbb{P}^2$ (i.e. ψ and φ^{-1} are both of type III)
- ii) The maps ψ and φ^{-1} are both of type I or both of type II, and share the same base-point.

Moreover, if $X_1 = \mathbb{F}_n$ and $X_0 = X_2 = \mathbb{F}_{n+1}$, for some $n \geq 1$, the map $\psi \circ \varphi$ is always an isomorphism.

Proof. If neither ψ nor φ^{-1} has a base-point, we have $X_1 = \mathbb{F}_1$ and $X_0 = X_2 = \mathbb{P}^2$, and both ψ and φ^{-1} are contractions of the (-1) -curve of \mathbb{F}_1 . This implies that $\psi \circ \varphi$ is an automorphism of \mathbb{P}^2 .

If ψ has a base-point, which is not a base-point of φ^{-1} , then $\psi \circ \varphi$ has a base-point, and is thus not an isomorphism. The same holds exchanging the role of ψ and φ .

The last case is when both ψ and φ^{-1} are links of type I or II, having the same base-point. The description of the links (Lemma 2.8) implies that $\psi \circ \varphi$ is an isomorphism. It remains to see that this is always the case when $X_1 = \mathbb{F}_n$ and $X_0 = X_2 = \mathbb{F}_{n+1}$. Indeed, this is true as the base-point of every elementary link from $(\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n})$ to $(\mathbb{F}_{n+1}, E_{n+1} \cup L_{\mathbb{F}_{n+1}})$ is the intersection point of E_n and $L_{\mathbb{F}_n}$. \square

Definition 2.13. A decomposition of a birational map $\varphi: (X, B) \dashrightarrow (X', B')$ into elementary links is called *reduced*, if the composition of two consecutive elementary links is never an automorphism.

Proposition 2.14. *Let $\varphi: (X, B) \dashrightarrow (X', B')$ be a birational map between two natural completions of \mathbb{A}^2 , and let $\varphi = \varphi_m \cdots \varphi_1$ be a reduced decomposition of φ into elementary links, with $m \geq 1$. Then, the following hold:*

- i) *The number l of base-points of φ is equal to the number of elementary links φ_i of type I or II. If $l \geq 1$, then φ has a unique proper base-point, which is equal to the base-point of φ_1 if φ_1 is of type I or II, or to the preimage by φ_1 of the base-point of φ_2 if φ_1 is of type III.*
- ii) *If $\varphi = \psi_k \cdots \psi_1$ is another decomposition into elementary links of φ , then $k \geq m$. Moreover, if $k = m$ the decomposition is unique, up to isomorphisms of completions.*

Proof. We prove i) by induction on m , the case $m = 1$ being obvious. Let $m \geq 2$.

Suppose that φ_1 is a link of type III. It contracts $E_1 \subset X$ onto a point which is not a base-point of φ_2 , and thus not of $\varphi_m \cdots \varphi_2$, by induction hypothesis. In consequence, the number of base-points of φ is equal to the one of $\varphi_m \cdots \varphi_2$, and φ has a unique proper base-point, being the preimage by φ_1 of the one of φ_2 .

Suppose now that φ_2 is a link of type III, which implies that φ_1 is a link $\mathbb{F}_2 \dashrightarrow \mathbb{F}_1$ of type II, with one base-point q . If $m = 2$, the result is clear. If $m \geq 3$, we use the induction hypothesis and see that the unique proper base-point of $\varphi_m \cdots \varphi_2$ is a point of $L_{\mathbb{F}_1} \setminus E_1$, which is not a base-point of $(\varphi_1)^{-1}$. This implies that φ has one more base-point as $\varphi_m \cdots \varphi_2$. Moreover, the proper base-point of $\varphi_m \cdots \varphi_2$ being on $L_{\mathbb{F}_1}$, contracted by $(\varphi_1)^{-1}$ onto q , this implies that q is the only proper base-point of φ .

To prove *i*), it remains to study the case where φ_1 and φ_2 are links of type I or II. As in the previous case, $\varphi_m \cdots \varphi_2$ has a unique proper base-point (here equal to the one of φ_2), which is not a base-point of $(\varphi_1)^{-1}$ but belongs to the curve contracted by $(\varphi_1)^{-1}$ onto the base-point of φ_1 . This again implies that φ has one more base-point as $\varphi_m \cdots \varphi_2$ and that its unique proper base-point is the one of φ_1 .

Now that *i*) is proved, let us show that it implies *ii*). Suppose the existence of another decomposition, that we can assume to be reduced, of length $k \leq m$: $\varphi = \varphi_m \cdots \varphi_1 = \psi_k \cdots \psi_1$. If φ has no base-point, both decomposition consist of one link of type III and the result is clear. Otherwise, the unique base-point of φ determines the first link, so we have $\psi_1 = \alpha \varphi_1$ for some isomorphism of natural completions α . Proceeding by induction, we get $k = m$ and the unicity of the decomposition. \square

Proposition 2.14 implies that the “length” of a reduced decomposition of φ only depends on φ . The following definition is thus natural.

Definition 2.15. The number of elementary links in a reduced decomposition of φ is called the *length* of φ and we denote it by $\text{len}(\varphi)$.

3. BIRATIONAL MAPS PRESERVING A CURVE AND THE PROOF OF THEOREM 1

This section is devoted to the proof of Theorem 1, which is done using elementary links and by looking at the singularities of the curve obtained in the natural completions.

Definition 3.1. If $\Gamma \subseteq X$ is a (closed) curve with no component in B and $\varphi: (X, B) \dashrightarrow (X, B)$ is a birational map, then we denote by $\varphi(\Gamma)$ the closure of $\varphi(\Gamma \cap X \setminus B)$ in X and call it the *image* of Γ under φ . Moreover, we denote by $\text{Bir}((X, B), \Gamma)$ the group of birational maps $\varphi: (X, B) \dashrightarrow (X, B)$ such that $\varphi(\Gamma) = \Gamma$.

Remark 3.2. If $\Gamma \subset X$ is a curve having no component in B , then $\text{Bir}((X, B), \Gamma) = \{g \in \text{Aut}(X \setminus B) \mid g(\Gamma \setminus B) = (\Gamma \setminus B)\}$. In particular, $\text{Bir}((X, B), \Gamma)$ corresponds to the group $\text{Aut}(\mathbb{A}^2, \Gamma \cap \mathbb{A}^2)$ of automorphisms of $X \setminus B = \mathbb{A}^2$ that preserve the closed curve $\Gamma \cap \mathbb{A}^2$ of \mathbb{A}^2 .

To study maps preserving a curve, we will study the singularities of the curves on the boundary.

Definition 3.3. Let Γ be a curve and let $p \in \Gamma$ be a $\bar{\mathbf{k}}$ -point. We define the *height* $\text{ht}_\Gamma(p)$ of Γ in p inductively, as follows

- i*) If Γ is smooth in p , then we define $\text{ht}_\Gamma(p) := 0$.
- ii*) Otherwise, let $\pi: \tilde{\Gamma} \rightarrow \Gamma$ be the blow-up of Γ in p and let p_1, \dots, p_n be the points of $\tilde{\Gamma}$ with $\pi(p_i) = p$. We define $\text{ht}_\Gamma(p) := \max_i \{ \text{ht}_{\tilde{\Gamma}}(p_i) + 1 \}$.

Recall that for any curves Γ_1, Γ_2 having no common component on a smooth projective surface, the intersection number $\Gamma_1 \cdot \Gamma_2$ is non-negative, and corresponds to the sum of the intersection numbers $(\Gamma_1 \cdot \Gamma_2)_p$, where p runs over all $\bar{\mathbf{k}}$ -points of $\Gamma_1 \cap \Gamma_2$. The intersection number $(\Gamma_1 \cdot \Gamma_2)_p$ satisfies $(\Gamma_1 \cdot \Gamma_2)_p \geq m_p(\Gamma_1) \cdot m_p(\Gamma_2)$, where $m_p(\Gamma_i)$ is the multiplicity of Γ_i at p , and equality holds if and only if the tangent cones of Γ_1 and Γ_2 at p are distinct.

The next proposition is the key ingredient in the proof of Theorem 1.

Proposition 3.4. *Let $\varphi: (X, B) \dashrightarrow (X', B')$ be a birational map between natural completions of \mathbb{A}^2 , admitting a reduced decomposition $\varphi = \varphi_m \cdots \varphi_1$ into elementary links, with $m \geq 1$. Let $\Gamma \subset X$ be a curve having no component in B and let $\Gamma' = \varphi(\Gamma) \subset X'$ be its image under φ . Suppose that one of the following holds:*

a) φ_1 is a link of type III (from \mathbb{F}_1 to \mathbb{P}^2) and $(E_1 \cdot \Gamma)_{p_0} > 1$, where $\{p_0\} := E_1 \cap L_{\mathbb{F}_1} =: A$.

b) φ_1 is a link of type I or II, and the finite set $A \subset L_X$ of $\bar{\mathbf{k}}$ -points lying on Γ which are not (proper) base-points of φ satisfies $\sum_{p \in A} (L_X \cdot \Gamma)_p > 1$.

Then, there exists a $\bar{\mathbf{k}}$ -point $q \in L_{X'}$ such that

$$\mathrm{ht}_{\Gamma'}(q) \geq l + \max_{p \in A} \mathrm{ht}_{\Gamma}(p),$$

where l is the number of base-points of φ^{-1} . If $l \geq 1$, then q can be chosen as the unique proper base-point of φ^{-1} , which is a \mathbf{k} -point.

Proof. We proceed by induction on the number m of elementary links in the reduced decomposition of φ . We distinguish the following cases, depending on the nature of the first link φ_1 .

i) If φ_1 is a link of type III, it contracts the curve $E_1 \subseteq \mathbb{F}_1$ onto a \mathbf{k} -point $q \in \mathbb{P}^2$. By assumption, we have $(E_1 \cdot \Gamma)_{p_0} > 1$. The multiplicity of $\Gamma_1 = \varphi_1(\Gamma)$ at q is equal to $E_1 \cdot \Gamma > 1$, which implies that Γ_1 is singular at q and that $\mathrm{ht}_{\Gamma_1}(q) \geq \mathrm{ht}_{\Gamma}(p_0) + 1$.

If $m = 1$, we have $l = 1$ and $\mathrm{ht}_{\Gamma_1}(q) \geq \mathrm{ht}_{\Gamma}(p) + 1 = l + \max_{p \in A} \mathrm{ht}_{\Gamma}(p)$ (because $A = \{p_0\}$), so we are done. So, assume $m > 1$. The point q is not a base point of φ_2 , since otherwise $\varphi_2 \circ \varphi_1$ would be an automorphism of \mathbb{F}_1 . Moreover, $(L_{\mathbb{P}^2} \cdot \Gamma_1)_q > 1$ because Γ_1 is singular at q . The claim follows by applying the induction hypothesis to $\varphi_m \cdots \varphi_2$, as q is not a base-point of $\varphi_m \cdots \varphi_2$.

ii) Suppose that φ_1 is a link of type II, i.e. $\varphi_1: \mathbb{F}_{n'} \dashrightarrow \mathbb{F}_n$ where $n' = n \pm 1$. By definition there exist blow-ups $\varepsilon: S \rightarrow \mathbb{F}_{n'}$ and $\eta: S \rightarrow \mathbb{F}_n$ of \mathbf{k} -points $q' \in L_{\mathbb{F}_{n'}}$ and $q \in L_{\mathbb{F}_n}$ respectively, such that $\varphi_1 = \eta \circ \varepsilon^{-1}$. Denote by $\tilde{\Gamma} \subseteq S$ the strict transform of Γ under ε^{-1} , and by $E_q \subset S$ the irreducible curve contracted by η onto q , which is the strict transform of $L_{\mathbb{F}_{n'}}$ under ε^{-1} . The map ε^{-1} is an isomorphism in a neighbourhood of every point of $A \subset L_{\mathbb{F}_{n'}}$. By hypothesis, one has $\sum_{p \in A} (L_{\mathbb{F}_{n'}} \cdot \Gamma)_p > 1$, which implies that $\tilde{\Gamma} \cdot E_q > 1$ on S . The curve $\Gamma_1 = \eta(\tilde{\Gamma}) \subset \mathbb{F}_n$ is thus singular at q , and the height of Γ_1 at q is at least equal to $1 + \max_{p \in A} \mathrm{ht}_{\Gamma}(p)$. If $m = 1$, then we are done. If $m > 1$, q is not a base point of φ_2 , since otherwise $\varphi_2 \circ \varphi_1$ would be an isomorphism. Moreover, q is a singular point of Γ_1 which belongs to $L_{\mathbb{F}_n}$, and if φ_2 is of type III, then $n = 1$, $n' = 2$, so q is the intersection point of E_1 and $L_{\mathbb{F}_1}$. In any case, the point q belongs to the set “ A ” associated to $\varphi_m \cdots \varphi_2$, so the result follows by induction.

iii) The last case is when φ_1 is of type I, i.e. when it is a map $\varphi_1: \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$. If $m = 1$, then $l = 0$ and the result is obvious, by choosing q as the image of a point of A with a maximal height. If $m > 1$, the link φ_2 is a map $\mathbb{F}_1 \dashrightarrow \mathbb{F}_2$ centered at the intersection point of E_1 and $L_{\mathbb{F}_1}$. In particular, the map φ_1 induces an isomorphism at a neighbourhood of any point of A , sending $L_{\mathbb{P}^2}$ onto $L_{\mathbb{F}_1}$, and sends the set A onto the set “ A ” associated to $\varphi_m \cdots \varphi_2$. The result follows by induction hypothesis. \square

The following corollary is a direct consequence of Proposition 3.4.

Corollary 3.5. *Let $\varphi \in \text{Bir}((X, B), \Gamma) \setminus \text{Aut}(X, B)$ and let $\varphi = \varphi_m \cdots \varphi_1$ be its reduced decomposition into elementary links. We assume that $(L_X \cdot \Gamma)_p > 1$ for some $\bar{\mathbf{k}}$ -point p and if φ_1 is of type III (i.e. a link from \mathbb{F}_1 to \mathbb{P}^2), then we also assume $\{p\} = E_1 \cap L_{\mathbb{F}_1}$.*

Then the height in Γ of the proper base point q of φ is (strictly) bigger than the height in Γ of every other point of $\Gamma \cap L_X$. In particular, $(L_X \cdot \Gamma)_q > 1$, φ and φ^{-1} have the same proper base-point q , and either $\Gamma \cap L_X = \{q\}$ or q is a singular point of Γ .

Proposition 3.6. *Let (X, B) be a natural completion of \mathbb{A}^2 and let Γ be a curve in X having no component in B . Then, there exists a natural completion (X', B') of \mathbb{A}^2 and a birational map $\varphi: (X, B) \dashrightarrow (X', B')$ such that one of the following holds:*

- 1) $\text{Bir}((X', B'), \varphi(\Gamma)) \subseteq \text{Aut}(X', B')$;
- 2) $\varphi(\Gamma)$ intersects transversally $L_{X'}$, i.e. $(\varphi(\Gamma) \cdot L_{X'})_p \leq 1$ for any $p \in X'(\bar{\mathbf{k}})$.

Proof. We define $(X_0, B_0) = (X, B)$, define $\varphi_0 = \text{id} \in \text{Aut}(X_0, B_0)$ and construct inductively a sequence of elementary links

$$(*) \quad (X_0, B_0) \xrightarrow{\varphi_1} (X_1, B_1) \xrightarrow{\varphi_2} (X_2, B_2) \xrightarrow{\varphi_3} \dots$$

Let $i \geq 0$ and assume that $\varphi_0, \dots, \varphi_i$ are already constructed. We write $\Gamma_i = \varphi_i \cdots \varphi_1(\Gamma)$, $G_i = (\text{Bir}(X_i, B_i), \Gamma_i)$ and define now φ_{i+1} in the following way:

- i) If $G_i \subseteq \text{Aut}(X_i, B_i)$ or Γ_i intersects transversally L_{X_i} , we define $(X_{i+1}, B_{i+1}) = (X_i, B_i)$ and $\varphi_{i+1} = \text{id}$.
- ii) If i) doesn't hold and the decomposition into elementary links of every $g \in G_i \setminus \text{Aut}(X_i, B_i)$ starts with the link $\tau_0: \mathbb{F}_1 \rightarrow \mathbb{P}^2$ (which contracts the (-1) -curve of \mathbb{F}_1), we define $\varphi_{i+1} = \tau_0$, and thus set $(X_{i+1}, B_{i+1}) = (\mathbb{P}^2, L_{\mathbb{P}^2})$.
- iii) If i) and ii) do not occur, there exists $g \in G_i \setminus \text{Aut}(X_i, B_i)$ such that the decomposition into elementary links of g starts with a link τ_1 being not from \mathbb{F}_1 to \mathbb{P}^2 . In this case we define $\varphi_{i+1} = \tau_1$, and (X_{i+1}, B_{i+1}) as the target of this link.

Now, we claim that the sequence $(*)$ satisfies the following two properties.

- a) If φ_{i+1} is not an automorphism, then we have for all $g \in G_i$

$$\text{len}(\varphi_{i+1} g \varphi_{i+1}^{-1}) < \text{len}(g) \quad \text{or} \quad \text{len}(\varphi_{i+1} g \varphi_{i+1}^{-1}) = \text{len}(g) = 0.$$

- b) The sequence $(*)$ is stationary after finitely many steps, i.e. φ_i is an automorphism for i large enough.

Property a) will serve to show Property b), which directly implies the result. It thus remains to prove these two properties.

Proof of Property a):

In case i), there is nothing to prove.

If we are in case ii), then for all $g \in G_i \setminus \text{Aut}(X_i, B_i)$ the decomposition into elementary links starts with $\tau_0: \mathbb{F}_1 \rightarrow \mathbb{P}^2$ and ends with $(\tau_0)^{-1}$, since G_i is a group. The conjugation of G_i by τ_0 decreases the length of any element of $G_i \setminus \text{Aut}(X_i, B_i)$ by two. Moreover, $(X_i, B_i) = (\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1})$, so $\text{Aut}(X_i, B_i)$ preserves the curve E_1 contracted by τ_0 , which implies that $\tau_0 \text{Aut}(X_i, B_i) (\tau_0)^{-1} \subset \text{Aut}(\mathbb{P}^2, L_{\mathbb{P}^2}) = \text{Aut}(X_{i+1}, B_{i+1})$. It follows that a) is satisfied.

Assume that we are in case iii). As we are not in case i) there exists $p_0 \in X_i$ such that $(\Gamma_i \cdot L_{X_i})_{p_0} > 1$. Moreover, by definition of case iii), there exists

$g \in G_i \setminus \text{Aut}(X_i, B_i)$ such that the decomposition into elementary links starts with $\varphi_{i+1}: (X_i, B_i) \dashrightarrow (X_{i+1}, B_{i+1})$, and φ_{i+1} has a base-point p since it does not go from \mathbb{F}_1 to \mathbb{P}^2 . Applying Corollary 3.5 to g , we see that $(\Gamma_i \cdot L_{X_i})_p > 1$ and that the height in Γ of p is bigger than the height in Γ of every other point of $\Gamma \cap L_X$. This implies that any element $h \in \text{Aut}(X_i, B_i) \cap G_i$ fixes p , and thus satisfies $\varphi_{i+1} h \varphi_{i+1}^{-1} \in \text{Aut}(X_{i+1}, B_{i+1})$. If $X_i = \mathbb{F}_1$, then φ_{i+1} is a link from \mathbb{F}_1 to \mathbb{F}_2 , so p is the intersection point $E_1 \cap L_{\mathbb{F}_1}$. We can thus apply Corollary 3.5 to any element $h \in G_i \setminus \text{Aut}(X_i, B_i)$, and see that p is the unique proper base-point of h , so the reduced decomposition of h starts with φ_{i+1} . This yields *a*).

Proof of Property b):

By property *a*), for all i , $\varphi_i \cdots \varphi_1$ is a reduced decomposition into elementary links, if we are for no index $< i$ in case *i*). For any $i \geq 0$ such that neither φ_{i+1} nor φ_{i+2} is an isomorphism, we define $p_i \in L_{X_i} \subset B_i \subset X_i$ as the unique proper base-point of $\varphi_{i+2} \circ \varphi_{i+1}$.

Under the assumption that neither of the three maps $\varphi_{i+1}, \varphi_{i+2}$ and φ_{i+3} is an isomorphism we show that $(\Gamma_i \cdot L_{X_i})_{p_i} \geq (\Gamma_{i+1} \cdot L_{X_{i+1}})_{p_{i+1}}$ and describe the cases when equality holds:

- φ_{i+1} is a link of type II: we can write $\varphi_{i+1} = \eta \circ \pi^{-1}$, where $\pi: S \rightarrow X_i$ is the blow-up of p_i and $\eta: S \rightarrow X_{i+1}$ is the blow-up of a point $q \neq p_{i+1}$. Denote by $\tilde{\Gamma}_i, \tilde{L}_{X_i} \subset S$ the strict transforms of Γ_i and L_{X_i} , and let E_{p_i} be the exceptional divisor of π . It follows that η contracts \tilde{L}_{X_i} . Note that η^{-1} restricts to an isomorphism in a neighbourhood of p_{i+1} , which sends $L_{X_{i+1}}$ onto E_{p_i} . This yields the following estimate

$$\begin{aligned} (\Gamma_i \cdot L_{X_i})_{p_i} &\geq m_{p_i}(\Gamma_i) \\ &= \tilde{\Gamma}_i \cdot E_{p_i} \\ &\geq (\tilde{\Gamma}_i \cdot E_{p_i})_{\eta^{-1}(p_{i+1})} \\ &= (\Gamma_{i+1} \cdot L_{X_{i+1}})_{p_{i+1}} \end{aligned}$$

where $m_{p_i}(\Gamma_i)$ denotes the multiplicity of Γ_i in p_i . Moreover, if we have $(\Gamma_i \cdot L_{X_i})_{p_i} = m_{p_i}(\Gamma_i) = (\Gamma_{i+1} \cdot L_{X_{i+1}})_{p_{i+1}} > 1$, then $\text{ht}_{\Gamma_i}(p_i) > \text{ht}_{\Gamma_{i+1}}(p_{i+1})$.

- φ_{i+1} is a link of type I: Remark that $(\varphi_{i+1})^{-1}: \mathbb{F}_1 \rightarrow \mathbb{P}^2$ is the blow-up of $p_i \in \mathbb{P}^2$ and that Γ_{i+1} and $L_{\mathbb{F}_1}$ are the strict transforms of Γ_i and $L_{\mathbb{P}^2}$ under $(\varphi_{i+1})^{-1}$ respectively. It follows that $(\Gamma_i \cdot L_{\mathbb{P}^2})_{p_i} \geq (\Gamma_{i+1} \cdot L_{\mathbb{F}_1})_{p_{i+1}}$.
- φ_{i+1} is a link of type III: Since φ_{i+1} is an isomorphism in a neighbourhood of p_i and $p_{i+1} = \varphi_{i+1}(p_i)$ it follows that $(\Gamma_i \cdot L_{\mathbb{F}_1})_{p_i} = (\Gamma_{i+1} \cdot L_{\mathbb{P}^2})_{p_{i+1}}$ and $\text{ht}_{\Gamma_i}(p_i) = \text{ht}_{\Gamma_{i+1}}(p_{i+1})$.

Now, assume towards a contradiction, that the sequence $(*)$ is never stationary, i.e. φ_i is not an isomorphism for all $i \geq 1$. According to this case-by-case-analysis, we see that $(\Gamma_i \cdot L_{X_i})_{p_i}$ is a decreasing sequence in i and for every $r > 1$ there are only finitely many i with $(\Gamma_i \cdot L_{X_i})_{p_i} = r$. Thus, there exists I such that for all $i \geq I$ we have $(\Gamma_i \cdot L_{X_i})_{p_i} \leq 1$. As the sequence $(*)$ is not stationary, we have a φ_{i+1} with $i \geq I$ which is a link of type I or II. The point p_i is thus the base-point of φ_{i+1} and satisfies $(\Gamma_i \cdot L_{X_i})_{p_i} \leq 1$. By Corollary 3.5 applied to an automorphism of $G_i \setminus \text{Aut}(X_i, B_i)$ that starts with φ_{i+1} , this implies that $(\Gamma_i \cdot L_{X_i})_q = 1$ for all $q \in L_{X_i} \cap \Gamma_i$. Hence, we are in case *i*), a contradiction. \square

Lemma 3.7. *Let $(X, B) = (\mathbb{P}^2, L_{\mathbb{P}^2})$, and let Γ be a conic (as always, reduced but not necessarily irreducible) in $X = \mathbb{P}^2$, intersecting $L_{\mathbb{P}^2}$ into two distinct points (not necessarily defined over \mathbf{k}).*

Then, $\text{Bir}((X, B), \Gamma) \subseteq \text{Aut}(X, B)$.

Proof. Suppose, for contradiction, the existence of $g \in \text{Bir}((X, B), \Gamma)$, which is not an automorphism of \mathbb{P}^2 .

Applying Proposition 3.4, one of the two points of $\Gamma \cap L_X$ is a base-point of g (and in particular it is defined over \mathbf{k}). By Proposition 2.9 and Lemma 2.12, there exists an integer $n \geq 2$ such that the reduced decomposition of g into elementary links starts with $\varphi_{2n}\varphi_{2n-1} \cdots \varphi_1$, where $\varphi_1: \mathbb{P}^2 \dashrightarrow \mathbb{F}_1$, $\varphi_{2n}: \mathbb{F}_1 \rightarrow \mathbb{P}^2$, $\varphi_i: \mathbb{F}_{i-1} \dashrightarrow \mathbb{F}_i$ and $\varphi_{i+n-1}: \mathbb{F}_{n-i+2} \dashrightarrow \mathbb{F}_{n-i+1}$ for $i = 2, \dots, n$. The curve $\varphi_1(\Gamma)$ intersects transversally E_1 into one point and $L_{\mathbb{F}_1}$ into one point, and does not pass through the intersection point of E_1 and $L_{\mathbb{F}_1}$, blown-up by φ_2 . This implies that the same holds for $\varphi_2\varphi_1(\Gamma)$, with E_2 and $L_{\mathbb{F}_2}$, and that $\varphi_2\varphi_1(\Gamma)$ passes through the point blown-up by $(\varphi_2)^{-1}$. Proceeding by induction, the curve $\varphi_n \cdots \varphi_1(\Gamma) \subset \mathbb{F}_n$ intersects transversally E_n into one point and $L_{\mathbb{F}_n}$ into one point, and passes through the point blown-up by $(\varphi_n)^{-1}$. In consequence, it does not pass through the point blown-up by φ_{n+1} , which implies that $\varphi_{n+1}\varphi_n \cdots \varphi_1(\Gamma) \subset \mathbb{F}_{n-1}$ intersects $E_{n-1} \cup L_{\mathbb{F}_{n-1}}$ into two distinct points, both being on E_{n-1} . Proceeding by induction, the curve $\varphi_{2n-1} \cdots \varphi_n \cdots \varphi_1(\Gamma) \subset \mathbb{F}_1$ intersects $E_1 \cup L_{\mathbb{F}_1}$ into two distinct points, both on E_1 . The curve $\varphi_{2n} \cdots \varphi_1(\Gamma) \subset \mathbb{P}^2$ intersects $L_{\mathbb{P}^2}$ into one point q , and is singular at this point, with two branches. The remaining part of the decomposition of g is equal to $\varphi_m \cdots \varphi_{2n+1}$, and its unique proper base-point is different from q . Proposition 3.4 implies that $g(\Gamma)$ is singular at a point of $B = L_{\mathbb{P}^2}$, which is a contradiction. \square

Proposition 3.8. *Let (X, B) be a natural completion of \mathbb{A}^2 and let Γ be a curve in X . If Γ intersects transversally L_X , there exists a natural completion (X', B') of \mathbb{A}^2 and a birational map $\varphi: (X, B) \dashrightarrow (X', B')$ such that one of the following holds:*

- i) $\text{Bir}((X', B'), \varphi(\Gamma)) \subseteq \text{Aut}(X', B')$;
- ii) $X' = \mathbb{P}^2$ and $\varphi(\Gamma) \subset \mathbb{P}^2$ is defined by a polynomial in $\mathbf{k}[x]$.

Proof. We can assume the existence of $g \in \text{Bir}((X, B), \Gamma) \setminus \text{Aut}(X, B)$, admitting a reduced decomposition $g = \varphi_m \cdots \varphi_1$ into elementary links.

a) Suppose that $X = \mathbb{P}^2$. By Corollary 3.5, the curve Γ intersects L_X into at most 2 points, hence Γ is a line or a conic. If Γ is a line the equation can be chosen to be $x = 0$, and if Γ is a conic, the result follows from Lemma 3.7.

b) Suppose that $X \not\cong \mathbb{P}^2$ but that there exists $1 \leq i \leq m$ such that φ_i is a link $\varphi_i: \mathbb{F}_1 \rightarrow \mathbb{P}^2$ of type III, contracting E_1 onto a point $q \in L_{\mathbb{P}^2}$. If $\Gamma' = \varphi_i \cdots \varphi_1(\Gamma)$ intersects transversally $L_{\mathbb{P}^2}$, we conclude by applying case a) to Γ' . There exists thus a point $p \in L_{\mathbb{P}^2}$ with $(\Gamma' \cdot L_{\mathbb{P}^2})_p > 1$. We claim that p can be chosen to be equal to q . If $i = 1$, this is true because $\Gamma' = \varphi_1(\Gamma)$ and Γ intersects transversally $L_{\mathbb{F}_1}$, which is the strict transform of $L_{\mathbb{P}^2}$. If $i > 1$, the claim follows from Proposition 3.4, applied to $(\varphi_1)^{-1} \cdots (\varphi_i)^{-1}: (\mathbb{P}^2, L_{\mathbb{P}^2}^2) \dashrightarrow (X, B)$. Since φ_{i+1} is a link of type I and q is not a base-point of φ_{i+1} , $\varphi_{i+1}(\Gamma')$ does not intersect transversally $L_{\mathbb{F}_1}$. This implies that $m \geq i + 2$, and that $(\varphi_m \cdots \varphi_{i+1})^{-1}$ has at least one base-point. Applying Proposition 3.4 to $\varphi_m \cdots \varphi_{i+1}$, the curve $\Gamma = \varphi_m \cdots \varphi_{i+1}(\Gamma')$ has a singular point

at the proper base-point of $(\varphi_m \cdots \varphi_{i+1})^{-1}$. Since this one lies on L_X , we get a contradiction.

c) We can now assume that $X = \mathbb{F}_n$ for some $n \geq 1$ and that all φ_i are links of type II. By Corollary 3.5, $L_{\mathbb{F}_n} \cap \Gamma$ contains at most 2 points and if it contains 2, then one of these is a base-point of φ_1 . If one of the points of $L_{\mathbb{F}_n} \cap \Gamma$ is the intersection point p_n of $L_{\mathbb{F}_n}$ and E_n , we perform an elementary link $\psi: \mathbb{F}_n \dashrightarrow \mathbb{F}_{n+1}$. Because $L_{\mathbb{F}_n} \cap \Gamma$ contains at most 2 points, the curve $\psi(\Gamma)$ intersects transversally $L_{\mathbb{F}_{n+1}}$, into at most 2 points. Moreover, writing $\{p_{n+1}\} = L_{\mathbb{F}_{n+1}} \cap E_{n+1}$, we obtain $(\psi(\Gamma) \cdot E_{n+1})_{p_{n+1}} = (\Gamma \cdot E_n)_{p_n} - 1$. Performing a sequence of elementary links if needed, we reduce to the case where $L_{\mathbb{F}_n} \cap \Gamma$ contains at most 2 points and that none of them belong to E_n .

If $L_{\mathbb{F}_n} \cap \Gamma$ is empty, then Γ is contained in a finite set of fibres of $\mathbb{F}_n \rightarrow \mathbb{P}^1$. Going from \mathbb{F}_n to \mathbb{F}_1 and then to \mathbb{P}^2 , we send the fibres onto lines of the form $x = a$ where $a \in \overline{\mathbf{k}}$, and see that $L_{\mathbb{F}_n}$ has equation in $\mathbf{k}[x]$.

It remains to see that it is not possible, in the case where $L_{\mathbb{F}_n} \cap \Gamma$ contains 1 or 2 points that belong to $L_{\mathbb{F}_n} \setminus E_n$, to have $g \in \text{Bir}((X, B), \Gamma) \setminus \text{Aut}(X, B)$ having a reduced decomposition consisting only of links of type II. Such an element has a decomposition $g = \varphi_{2k} \cdots \varphi_1$, where $k \geq 1$, φ_i is a link $\mathbb{F}_{n+i-1} \dashrightarrow \mathbb{F}_{n+i}$ for $i = 1, \dots, k$, φ_i is a link $\mathbb{F}_{n+2k-i+1} \dashrightarrow \mathbb{F}_{n+2k-i}$ for $i = k+1, \dots, 2k$, and where $(\varphi_k)^{-1}$ and φ_{k+1} do not have the same base-point (see Lemma 2.12). In particular, the base-point of φ_1 is the intersection point of E_n and $L_{\mathbb{F}_n}$, and does not lie on Γ , so $\varphi_1(\Gamma)$ passes through the base-point of $(\varphi_1)^{-1}$, which is not a base-point of φ_2 . By induction, we deduce that $(\varphi_i \cdots \varphi_1)(\Gamma)$ passes through the base-point of $(\varphi_i)^{-1}$, for $i = 1, \dots, 2k$. In the case $i = 2k$, this implies that $g(\Gamma)$ passes through the base-point of $(\varphi_{2k})^{-1}$, i.e. through the intersection point of E_n and $L_{\mathbb{F}_n}$, contradicting the fact that $g(\Gamma) = \Gamma$. \square

Corollary 3.9. *Let (X, B) be a natural completion of \mathbb{A}^2 and let Γ be a curve in X having no component in B . Then, there exists a natural completion (X', B') of \mathbb{A}^2 and a birational map $\varphi: (X, B) \dashrightarrow (X', B')$ such that one of the following holds:*

- 1) $\text{Bir}((X', B'), \varphi(\Gamma)) \subseteq \text{Aut}(X', B')$;
- 2) $X' = \mathbb{P}^2$ and $\varphi(\Gamma) \subset \mathbb{P}^2$ is defined by a polynomial in $\mathbf{k}[x]$.

Proof. Follows directly from Propositions 3.6 and 3.8. \square

Theorem 1 is now a direct consequence of Corollary 3.9:

Proof of Theorem 1. In the case where the equation of Γ is in $\mathbf{k}[x]$, i.e. when Γ is a fence, the explicit description of $\text{Aut}(\mathbb{A}^2, \Gamma)$ is an easy calculation. If Γ is not equivalent to a fence by an automorphism of \mathbb{A}^2 , Corollary 3.9 and Lemma 2.6 imply that we can conjugate $\text{Aut}(\mathbb{A}^2, \Gamma)$ to a subgroup of $\text{Aff}(\mathbb{A}^2)$ or J_n for some n . This implies that $\text{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group. Moreover, we obtain a morphism of algebraic groups $\text{Aut}(\mathbb{A}^2, \Gamma) \rightarrow \text{Aut}(\Gamma)$. It remains to observe that curves fixed pointwise by elements of J_n and $\text{Aff}(\mathbb{A}^2)$ are fences in a suitable coordinate system of \mathbb{A}^2 . \square

3.1. Generalisation to other subsets.

Definition 3.10. If $\Delta \subset \mathbb{A}^2(\overline{\mathbf{k}})$ is any subset, we denote by $\text{Aut}(\mathbb{A}^2, \Delta)$ the group of automorphisms of \mathbb{A}^2 that leave the set Δ invariant, and denote by $\text{Aut}_F(\mathbb{A}^2, \Delta)$ the group of automorphisms of \mathbb{A}^2 that fix any element of Δ .

By Definition, $\text{Aut}_F(\mathbb{A}^2, \Delta)$ is always a normal subgroup of $\text{Aut}(\mathbb{A}^2, \Delta)$. Moreover, if $\text{Aut}(\mathbb{A}^2, \Delta)$ is an algebraic group, then $\text{Aut}_F(\mathbb{A}^2, \Delta)$ is an algebraic subgroup. Theorem 1 implies the following result:

Proposition 3.11. *Let $\Delta \subset \mathbb{A}^2(\overline{\mathbf{k}})$ be a subset. Applying an element $\varphi \in \text{Aut}(\mathbb{A}^2)$, the following hold:*

- (i) *The set Δ is contained in a fence given by $f(x) = 0$ where $f \in \mathbf{k}[x]$, and the group $\text{Aut}_F(\mathbb{A}^2, \Delta)$ is not algebraic: it contains the group*

$$\{(x, y) \mapsto (x, y + p(x)f(x)) \mid p \in \mathbf{k}[x]\} \simeq \mathbf{k}[x].$$

- (ii) *The group $\text{Aut}(\mathbb{A}^2, \Delta)$ is equal to $\{g \in \text{Aff}(\mathbb{A}^2) \mid g(\Delta) = \Delta\}$, and then $\text{Aut}(\mathbb{A}^2, \Delta)$ is an algebraic subgroup of $\text{Aff}(\mathbb{A}^2)$. Moreover, $\text{Aut}_F(\mathbb{A}^2, \Delta)$ is trivial.*
- (iii) *There exists an integer $n \geq 1$ such that $\text{Aut}(\mathbb{A}^2, \Delta)$ is equal to $\{g \in J_n \mid g(\Delta) = \Delta\}$, and then $\text{Aut}(\mathbb{A}^2, \Delta)$ is an algebraic subgroup of J_n . Moreover, $\text{Aut}_F(\mathbb{A}^2, \Delta)$ is trivial.*

Proof. Denote by $I(\Delta) \subset \mathbf{k}[x, y]$ the ideal of polynomials vanishing on Δ , and by $\overline{\Delta} \subset \mathbb{A}^2(\overline{\mathbf{k}})$ the closure of Δ , which is the set of points where $I(\Delta)$ vanishes. Then we obtain $\text{Aut}(\mathbb{A}^2, \Delta) = \text{Aut}(\mathbb{A}^2, \overline{\Delta})$ and $\text{Aut}_F(\mathbb{A}^2, \Delta) = \text{Aut}_F(\mathbb{A}^2, \overline{\Delta})$. We can thus replace Δ with $\overline{\Delta}$.

If Δ is a finite union of points, we get case (i). Otherwise, Δ is the union of one curve Γ (reduced but not necessarily irreducible) and of a finite number of points of $\mathbb{A}^2(\overline{\mathbf{k}})$, and $\text{Aut}(\mathbb{A}^2, \Delta) \subset \text{Aut}(\mathbb{A}^2, \Gamma)$. The result follows then from the description of Γ and $\text{Aut}(\mathbb{A}^2, \Gamma)$, given in Theorem 1. \square

In the case where Δ is finite, the group $\text{Aut}(\mathbb{A}^2, \Delta)$ is quite big; indeed it is often maximal. This is the case for example when $\mathbf{k} = \mathbb{C}$, as pointed out to us by J.-P. Furter and P.-M. Poloni. This is a consequence of the following observation.

Lemma 3.12. *Let G be a group acting on a set S . Let $\Delta \subset S$ be a finite subset of $r \geq 1$ points. Suppose that G acts $2r$ -transitively on S , and that $|S| > 2r$. Then,*

$$G_\Delta = \{g \in G \mid g(\Delta) = \Delta\}$$

is a maximal subgroup of G .

Proof. Since G_Δ is not trivial and not equal to G (because of r -transitivity), it suffices to take $a \in G \setminus G_\Delta$ and to show that a and G_Δ generate G . We can write $\Delta = \{x_1, \dots, x_r\}$, with $a(x_1), \dots, a(x_k) \notin \Delta$ and $a(x_{k+1}), \dots, a(x_r) \in \Delta$, where $1 \leq k \leq r$. The hypotheses yield the existence of $g \in G$ that fixes $x_1, \dots, x_r, a(x_2), \dots, a(x_k)$, and does not fix $a(x_1)$. Then, $g \in G_\Delta$ and $f = a^{-1}ga$ fixes x_2, \dots, x_r but $f(x_1) \notin \Delta$.

It remains to see that any $h \in G \setminus G_\Delta$ is generated by f and G_Δ . We write $\Delta = \{z_1, \dots, z_r\}$, with $h(z_1), \dots, h(z_j) \notin \Delta$ and $h(z_{j+1}), \dots, h(z_r) \in \Delta$, where $1 \leq j \leq r$. Replacing h with its composition with an element of G_Δ we can assume that $h(z_i) = z_i$ for $i = j+1, \dots, r$. For $i = 1, \dots, j$, we choose $g_i \in G_\Delta$ that sends z_i onto x_1 and sends $h(z_i)$ onto $f(x_1)$. Then, $(g_i)^{-1}fg_i$ sends z_i onto $h(z_i)$ and fixes $\Delta \setminus \{z_i\}$. Composing this element with an element of G_Δ , we find an element f_i , generated by f and G_Δ , which sends z_i onto $h(z_i)$ and fixes $(\Delta \cup h(\Delta)) \setminus \{z_i, h(z_i)\}$.

Since $h^{-1}f_1 \cdots f_j$ belongs to G_Δ , this achieves the proof. \square

Corollary 3.13. *Assume that the ground field \mathbf{k} is not a finite field of characteristic 2 and let $\Delta \subset \mathbb{A}^n(\mathbf{k})$ be a finite proper non-empty set, with $n \geq 2$. Then, $\text{Aut}(\mathbb{A}^n, \Delta)$ is a maximal subgroup of $\text{Aut}(\mathbb{A}^n)$.*

Proof. If \mathbf{k} is infinite, we use Lemma 3.12 and the fact that $\text{Aut}(\mathbb{A}^2)$ acts m -transitively on $\mathbb{A}^2(\mathbf{k})$ for every $m \geq 1$, which can be seen using the subgroup

$$\{(x_1, \dots, x_n) \mapsto (x_1 + p(x_2, \dots, x_n), x_2, \dots, x_n) \mid p \in \mathbf{k}[x_2, \dots, x_n]\},$$

and permutations of coordinates.

If \mathbf{k} is a finite field of characteristic > 2 , the group $\text{Aut}(\mathbb{A}^2)$ acts m -transitively on $\mathbb{A}^2(\mathbf{k})$ for each m by [Mau01]; we can then apply Lemma 3.12 to Δ or its complement. \square

Corollary 3.14. *Assume that the ground field \mathbf{k} is a finite field of characteristic 2 and let $\Delta \subset \mathbb{A}^n(\mathbf{k})$ be a finite proper non-empty set, with $n \geq 2$.*

Then, the group $\text{Aut}(\mathbb{A}^n, \Delta)$ is a maximal subgroup of $\text{Aut}(\mathbb{A}^n)$ if and only if $|\Delta| \neq \frac{1}{2}|\mathbb{A}^n(\mathbf{k})|$.

Proof. Let us write $|\mathbb{A}^n(\mathbf{k})| = 2m$ for some integer m .

If $|\Delta| < m$, the fact that $\text{Aut}(\mathbb{A}^n, \Delta)$ is a maximal subgroup of $\text{Aut}(\mathbb{A}^n)$ follows from Lemma 3.12 and from the fact that the action of $\text{Aut}(\mathbb{A}^n)$ on the $2m$ points of $\mathbb{A}^n(\mathbf{k})$ give all even permutations (see [Mau01]), and thus acts $(2m-2)$ -transitively. If $|\Delta| > m$, we exchange Δ with its complement.

If $|\Delta| = m$, there exists an automorphism φ of \mathbb{A}^n exchanging Δ with its complement (by the result of [Mau01] cited before). Denoting by H the group generated by $\text{Aut}(\mathbb{A}^n, \Delta)$ and φ , we have $\text{Aut}(\mathbb{A}^n, \Delta) \subsetneq H \subsetneq \text{Aut}(\mathbb{A}^n)$. \square

Remark 3.15. Corollaries 3.13 and 3.14 raise the question of describing all maximal subgroups of $\text{Aut}(\mathbb{A}^n)$ in general.

3.2. Generalisation to higher dimension. Let us show that in higher dimension, the hypersurfaces $X \subset \mathbb{A}^n$ such that $\text{Aut}(\mathbb{A}^n, X)$ is not an algebraic group are not as simple as in dimension 2:

Example 3.16. Let $X \subset \mathbb{A}^3$ be the hypersurface with equation $xy = f(z)$, for some polynomial $f \in \mathbf{k}[z]$. Then, $\text{Aut}(\mathbb{A}^3, X)$ contains the group

$$\left\{ (x, y, z) \mapsto \left(x, y + \frac{f(z + xq(x)) - f(z)}{x}, z + xq(x) \right) \mid q \in \mathbf{k}[x] \right\} \simeq \mathbf{k}[x]$$

and thus it is not an algebraic group.

A possible generalisation of Theorem 1 would be to show that every hypersurface $X \subset \mathbb{A}^3$ such that $\text{Aut}(\mathbb{A}^3, X)$ is not an algebraic group admits an \mathbb{A}^1 -fibration $X \rightarrow \mathbb{A}^1$ given by a coordinate projection.

4. CLASSIFICATION OF THE POSSIBLE GROUP ACTIONS AND THE PROOF OF THEOREM 2

This section is devoted to the proof of Theorem 2, which describes more precisely the curves and groups appearing in Theorem 1, in the case where the ground field \mathbf{k} is perfect, and where the curve is geometrically irreducible.

4.1. The possibilities for Γ and $\text{Aut}(\mathbb{A}^2, \Gamma)$.

Lemma 4.1. *Let Γ be an affine geometrically irreducible curve, defined over a perfect field \mathbf{k} . The group $\text{Aut}(\Gamma)$ is an affine algebraic group. If it has positive dimension, one of the following holds:*

- i) $\Gamma \simeq \mathbb{A}^1$;
- ii) Γ is a unicuspidal curve with normalization \mathbb{A}^1 ;
- iii) Γ is isomorphic, over $\overline{\mathbf{k}}$, to $\mathbb{A}^1 \setminus \{0\}$.

Proof. Let $\tilde{\Gamma}$ be the normalization of Γ , which can be viewed as an open subset of a smooth projective curve X , defined over \mathbf{k} . We write $(X \setminus \tilde{\Gamma})(\overline{\mathbf{k}}) = \{x_1, \dots, x_r\}$ its complement, which is a finite set of points x_1, \dots, x_r , not necessarily all defined over \mathbf{k} (the union is however invariant by $\text{Gal}(\overline{\mathbf{k}}/\mathbf{k})$). We also denote by x_{r+1}, \dots, x_m the $\overline{\mathbf{k}}$ -points of $\tilde{\Gamma}$ which are sent onto the singular $\overline{\mathbf{k}}$ -points of Γ . As before, not all are necessarily defined over \mathbf{k} .

This yields a natural inclusion

$$\text{Aut}(\Gamma) \subseteq \{g \in \text{Aut}(X) \mid g(\{x_1, \dots, x_m\}) = \{x_1, \dots, x_m\}\},$$

and a group homomorphism $\text{Aut}(\Gamma) \rightarrow \text{Sym}_m$. The kernel is of the same dimension as $\text{Aut}(\Gamma)$.

Let us recall easy classical facts on automorphisms of smooth projective curves. If the genus of X is at least 2, then $\text{Aut}(X)$ is finite. If the genus is 1, the subgroup of $\text{Aut}(X)$ that fixes a point is also finite. If X is rational, the subgroup of automorphisms fixing three points is trivial.

If the dimension of $\text{Aut}(X)$ is positive, we obtain that X is rational and that $1 \leq r \leq m \leq 2$.

If $r = m = 1$, then $\Gamma = \tilde{\Gamma} \simeq \mathbb{A}^1$ (every form of the affine line over a perfect field is trivial, see [Rus70]).

If $r = 1$ and $m = 2$, then $\tilde{\Gamma} \simeq \mathbb{A}^1$ and Γ is a unicuspidal curve.

If $r = m = 2$, then $\Gamma = \tilde{\Gamma}$ is smooth and isomorphic, over $\overline{\mathbf{k}}$, to $\mathbb{A}^1 \setminus \{0\}$. \square

Remark 4.2. Lemma 4.1 is false over a non-perfect field, since there are non-trivial forms of the affine line and its additive group; see [Rus70] for a classification of such curves.

Lemma 4.3. *Assume that \mathbf{k} is a perfect field. Let $\Gamma \subseteq \mathbb{A}^2$ be a closed geometrically irreducible curve and assume that $\text{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group that is of positive dimension (over $\overline{\mathbf{k}}$). Then, $\text{Aut}(\mathbb{A}^2, \Gamma)$ contains a closed subgroup G defined over \mathbf{k} , which is either*

- i) a one-dimensional torus, i.e. isomorphic to \mathbb{G}_m over $\overline{\mathbf{k}}$,
- ii) or isomorphic to \mathbb{G}_a over \mathbf{k} .

Proof. The algebraic group $\text{Aut}(\mathbb{A}^2, \Gamma)$ is isomorphic to a closed subgroup of $\text{Aut}(\Gamma)$ (Theorem 1). This gives three possibilities for Γ , according to Lemma 4.1.

(i) $\Gamma \simeq \mathbb{A}^1$, hence $\text{Aut}(\Gamma) \simeq \mathbb{G}_a \rtimes \mathbb{G}_m$, and contains thus a closed torus \mathbb{G}_m .

(ii) Γ is a unicuspidal curve, in which case $\text{Aut}(\Gamma)$ is a torus.

(iii) Γ is isomorphic, over $\overline{\mathbf{k}}$, to $\mathbb{A}^1 \setminus \{0\}$. The connected component of the identity $\text{Aut}(\mathbb{A}^2, \Gamma)^0$ is a connected algebraic group, defined over \mathbf{k} , which is a torus. \square

The two possibilities given by Lemma 4.3 are described respectively in §4.2 and §4.3. We will in particular show that the second case does not occur.

4.2. Torus actions.

Lemma 4.4. *Assume that \mathbf{k} is a perfect field. Let $\Gamma \subseteq \mathbb{A}^2$ be a closed geometrically irreducible curve and assume that $\text{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group that contains a closed one-dimensional torus T . Then there exists an automorphism $\varphi: \mathbb{A}^2 \rightarrow \mathbb{A}^2$ such that $\varphi \circ T \circ \varphi^{-1} \subseteq \text{GL}(2, \mathbf{k})$.*

Proof. By Corollary 3.9 we can assume that either $\text{Aut}(\mathbb{A}^2, \Gamma) \subseteq \text{Aut}(\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n})$ or $\text{Aut}(\mathbb{A}^2, \Gamma) \subseteq \text{Aut}(\mathbb{P}^2, L_{\mathbb{P}^2})$.

Moreover, we can assume that T is defined over \mathbf{k} and T is isomorphic to \mathbb{G}_m over $\overline{\mathbf{k}}$ (Lemma 4.3).

Assume $\text{Aut}(\mathbb{A}^2, \Gamma) \subseteq \text{Aut}(\mathbb{F}_n, E_n \cup L_{\mathbb{F}_n})$. Thus T acts on $L_{\mathbb{F}_n}$. If this action is trivial, then there exists a fixed point on $L_{\mathbb{F}_n} \setminus E_n$ that is defined over \mathbf{k} . Otherwise, T has exactly two fixed points on $L_{\mathbb{F}_n} \simeq \mathbb{P}^1$ defined over $\overline{\mathbf{k}}$. As $L_{\mathbb{F}_n} \cap E_n$ is a fixed point of the T -action that is defined over \mathbf{k} , there is a fixed point on $L_{\mathbb{F}_n} \setminus E_n$ that is defined over \mathbf{k} . Thus by performing elementary links, we can assume that $T \subseteq \text{Aut}(\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1})$. But T preserves the exceptional divisor E_1 and therefore $\varphi \circ T \circ \varphi^{-1} \subseteq \text{Aut}(\mathbb{P}^2, L_{\mathbb{P}^2})$, where $\varphi: \mathbb{F}_1 \rightarrow \mathbb{P}^2$ denotes the contraction of E_1 onto a point in $L_{\mathbb{P}^2}$.

Thus we are left over with the case $T \subseteq \text{Aut}(\mathbb{P}^2, L_{\mathbb{P}^2})$. It is enough to show that the induced action of T on $\mathbb{A}^2 = \mathbb{P}^2 \setminus L_{\mathbb{P}^2}$ has a fixed point that is defined over \mathbf{k} . The set of points of $\mathbb{A}^2(\overline{\mathbf{k}})$ that are fixed by $T(\overline{\mathbf{k}})$ consists either of one affine line or one point. This set is invariant by the action of the Galois group $\text{Gal}(\overline{\mathbf{k}}/\mathbf{k})$, and is thus defined over \mathbf{k} . Using again the fact that every form of the affine line over a perfect field is trivial (see [Rus70]), we find a \mathbf{k} -point of \mathbb{A}^2 fixed by T . \square

Proposition 4.5. *Assume that \mathbf{k} is perfect. Let $\Gamma \subseteq \mathbb{A}^2$ be a closed geometrically irreducible curve and assume that $\text{Aut}(\mathbb{A}^2, \Gamma)$ contains a closed one-dimensional torus. After conjugation by an automorphism of \mathbb{A}^2 , the curve Γ*

- i) *has equation $x = 0$, or*
- ii) *has equation $x^b = \lambda y^a$ where $a, b > 1$ are coprime integers and $\lambda \in \mathbf{k}^*$, or*
- iii) *has equation $x^b y^a = \lambda$ where $a, b \geq 1$ are coprime integers and $\lambda \in \mathbf{k}^*$, or*
- iv) *has equation $\lambda x^2 + \nu y^2 = 1$, where $\lambda, \nu \in \mathbf{k}^*$, $-\lambda\nu$ is not a square in \mathbf{k} and $\text{char}(\mathbf{k}) \neq 2$, or*
- v) *has equation $x^2 + \mu xy + y^2 = 1$, where $\mu \in \mathbf{k}^*$, $x^2 + \mu x + 1$ has no root in \mathbf{k} and $\text{char}(\mathbf{k}) = 2$.*

Moreover, the group $\text{Aut}(\mathbb{A}^2, \Gamma)$ is respectively equal to

- i) $\{(x, y) \mapsto (ax, by + P(x)) \mid a, b \in \mathbf{k}^*, P \in \mathbf{k}[x]\} \simeq \mathbf{k}[x] \rtimes (\mathbf{k}^*)^2$;
- ii) $\{(x, y) \mapsto (t^a x, t^b y) \mid t \in \mathbf{k}^*\} \simeq \mathbf{k}^*$;
- iii) $\{(x, y) \mapsto (t^a x, t^{-b} y) \mid t \in \mathbf{k}^*\} \simeq \mathbf{k}^*$ if $(a, b) \neq (1, 1)$;
 $\{(x, y) \mapsto (tx, t^{-1}y) \mid t \in \mathbf{k}^*\} \cup \{(x, y) \mapsto (ty, t^{-1}x) \mid t \in \mathbf{k}^*\} \simeq \mathbf{k}^* \rtimes \mathbb{Z}/2\mathbb{Z}$ if $(a, b) = (1, 1)$;
- iv) $T \rtimes \langle \sigma \rangle \simeq T \rtimes \mathbb{Z}/2\mathbb{Z}$, where $T, \{\sigma\} \subset \text{GL}(2, \mathbf{k})$ are given by

$$T = \left\{ \begin{pmatrix} a & -\nu b \\ \lambda b & a \end{pmatrix} \mid a^2 + \lambda \nu b^2 = 1 \right\}, \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover T is a torus, which is not \mathbf{k} -split.

- v) $T \rtimes \langle \sigma \rangle \simeq T \rtimes \mathbb{Z}/2\mathbb{Z}$, where $T, \{\sigma\} \subset \text{GL}(2, \mathbf{k})$ are given by

$$T = \left\{ \begin{pmatrix} a & b \\ b & a + \mu b \end{pmatrix} \mid a^2 + \mu ab + b^2 = 1 \right\}, \sigma = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}.$$

Moreover T is a torus which is isomorphic to Γ (and which is not \mathbf{k} -split).

Proof. By Theorem 1, we can suppose that $\text{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group, and using Lemma 4.4, we can moreover assume that Γ is preserved by a torus $T \subset \text{GL}(2, \mathbf{k})$.

There exists an element $\psi \in \text{GL}(2, \overline{\mathbf{k}})$ which conjugates $T(\overline{\mathbf{k}})$ to

$$\lambda \mapsto \begin{pmatrix} \lambda^a & 0 \\ 0 & \lambda^b \end{pmatrix}$$

for integers a, b with $(a, b) \neq (0, 0)$ and a, b are coprime. If a or b is equal to zero, then it follows that Γ is a line. Hence we can assume that a and b are non-zero. Now, let $(x_0, y_0) \in \psi(\Gamma(\overline{\mathbf{k}}))$, such that $(x_0, y_0) \neq (0, 0)$. If x_0 or y_0 is zero, then Γ is again a line. Hence, we may assume that $x_0 \neq 0 \neq y_0$. By symmetry, we can assume that $b > 0$. Then the equation of $\psi(\Gamma)$ is

$$\begin{aligned} y_0^a x^b - x_0^b y^a &= 0 & \text{if } a > 0, \\ y_0^a x^b y^{-a} - x_0^b &= 0 & \text{if } a < 0. \end{aligned}$$

In the first case, we can assume that $a > 1$ or $b > 1$, otherwise the curve is a line. This implies that $a \neq b$, since both are coprime, and we can thus assume that $a > b$. The Galois group $\text{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ fixes the unique point at infinity, which is then defined over \mathbf{k} . Hence, we can assume that the unique point of the closure of Γ at infinity is the direction $y = 0$, and that the equation of Γ is the polynomial $(\alpha x + \beta y)^b - (\gamma y)^a \in \mathbf{k}[x, y]$, for some $\alpha, \beta, \gamma \in \overline{\mathbf{k}}$, $\alpha\gamma \neq 0$. If $\beta/\alpha \in \mathbf{k}$, we make a change of coordinates $(x, y) \mapsto (x - \frac{\beta}{\alpha}y, y)$ and obtain an equation of the form $x^b - \lambda y^a$ for some $\lambda \in \mathbf{k}$, as desired. It remains to see that β/α always belong to \mathbf{k} . If the characteristic of \mathbf{k} is zero, we develop $(\alpha x + \beta y)^b$ and divide the coefficient of $x^{b-1}y$ by the coefficient of x^b . If the characteristic of \mathbf{k} is $p > 0$, we write $b = qm$, where q is a power of p and p does not divide m . Developing, we find

$$(\alpha x + \beta y)^b = (\alpha^q x^q + \beta^q y^q)^m = \alpha^b x^b + m\alpha^{b-q}\beta^q x^{b-q}y^q + \dots$$

hence $(\beta/\alpha)^q \in \mathbf{k}$, which implies, since \mathbf{k} is a perfect field, that $\beta/\alpha \in \mathbf{k}$.

In the second case ($a < 0$), the closure of the curve $\psi(\Gamma)$ has two points at infinity in \mathbb{P}^2 . If $a \neq -b$, the two points have different multiplicities. In consequence the Galois group $\text{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ has to fix the two points at infinity of Γ , which are thus defined over \mathbf{k} . We can assume that these points correspond to the directions $x = 0$ and $y = 0$, and that $\psi \in \text{GL}(2, \overline{\mathbf{k}})$ is diagonal. The equation of Γ is then of the form $x^b y^a - \lambda$ for some $\lambda \in \mathbf{k}^*$, and we get case *iii*). The only remaining case is when $(-a, b) = (1, 1)$ and the two points at infinity of Γ are exchanged by $\text{Gal}(\overline{\mathbf{k}}/\mathbf{k})$. The equation of $\psi(\Gamma)$ being of the form $xy = x_0 y_0$, the equation of Γ is of the form

$$\lambda x^2 + \mu xy + \nu y^2 = 1,$$

where $\lambda, \mu, \nu \in \mathbf{k}$, and $\lambda x^2 + \mu xy + \nu y^2 \in \mathbf{k}[x, y]$ is irreducible. When the characteristic of \mathbf{k} is not 2, we can make a change of coordinates $(x, y) \mapsto (x - \frac{\mu}{2\lambda}y, y)$ and assume that $\mu = 0$. The two points at infinity are thus given by $\lambda x^2 + \nu y^2 = 0$. Because the two points are not defined over \mathbf{k} , we find that $-\nu\lambda$ is not a square in \mathbf{k} , and obtain *iv*). If the characteristic of \mathbf{k} is 2, the elements λ, ν are squares in \mathbf{k} , since \mathbf{k} is perfect. Making a diagonal change of variables, we can then assume that $\lambda = \nu = 1$. Then $x^2 + \mu x + 1$ has no root in \mathbf{k} , and we get *v*).

It remains to prove that $\text{Aut}(\mathbb{A}^2, \Gamma)$ has the desired form. Case *i*) is a direct calculation. In cases *ii*), *iii*), *iv*), *v*), it can be checked that the group given are

contained in $\text{Aut}(\mathbb{A}^2, \Gamma)$, so we just need to see that there is no other automorphism. The group homomorphism $\text{Aut}(\mathbb{A}^2, \Gamma) \rightarrow \text{Aut}(\Gamma)$ being injective, the only case to consider is the curve $x^b y^a - \lambda = 0$, with $(a, b) \neq (1, 1)$, and to prove that there is no automorphism of \mathbb{A}^2 inducing on Γ an “exchange” of the two points at infinity. These two points are $p_1 = (1 : 0 : 0) \in \mathbb{P}^2$ and $p_2 = (0 : 1 : 0) \in \mathbb{P}^2$, and have multiplicity a and b respectively on Γ , and $\text{ht}_\Gamma(p_1) = \text{ht}_\Gamma(p_2) > 0$ if $b > 1$. We can moreover assume $a > b \geq 1$. The hypothetic automorphism extends to a birational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ which is not an automorphism of \mathbb{P}^2 , and thus decomposes into a sequence of elementary links $\varphi = \varphi_m \cdots \varphi_1$ (Proposition 2.9). By Corollary 3.5, we have $b = 1$ and the point blown-up by the first link $\varphi_1 : (\mathbb{P}^2, L_{\mathbb{P}^2}) \dashrightarrow (\mathbb{F}_1, E_1 \cup L_{\mathbb{F}_1})$ is p_1 . Looking at the equation of Γ in \mathbb{A}^2 , we can describe the closure of Γ on \mathbb{F}_1 . This projective curve is smooth, intersects transversally $L_{\mathbb{F}_1}$ in one point away from E_1 , corresponding to p_2 , and intersects E_1 into one point q_1 , with multiplicity a , corresponding to p_1 ; this latter is moreover not on $L_{\mathbb{F}_1}$. In consequence, the next links of type II do not affect the point q_1 , and after the first link of type III, the image of the curve is singular at a point of \mathbb{P}^2 with multiplicity $\geq a$ and height 1. This point being not the base-point of the next elementary links, the image of Γ by φ has again a singular point, corresponding to the image of p_1 . It is thus not possible to “exchange” p_1 and p_2 . \square

4.3. \mathbb{G}_a -actions. The classification of all \mathbb{G}_a -actions on \mathbb{A}^2 is known when the ground field \mathbf{k} is of characteristic 0 [Ren68] or algebraically closed of positive characteristic [Miy71]. The following lemma gives the generalisation of MIYANISHI’s result to the case where \mathbf{k} is perfect. The proof is probably known to the specialists, we include it for the sake of completeness and lack of reference.

Proposition 4.6. *Assume that \mathbf{k} is a perfect field of characteristic $p > 0$. Then every \mathbb{G}_a -action on \mathbb{A}^2 that is defined over \mathbf{k} has the form*

$$(t, x, y) \mapsto (x, y + t f_0(x) + t^p f_1(x) + \dots + t^{p^n} f_n(x)).$$

Proof. By [Miy71] it follows that the \mathbb{G}_a -action has the claimed form over the algebraic closure $\overline{\mathbf{k}}$. Thus there exists a \mathbb{G}_a -invariant polynomial $f \in \overline{\mathbf{k}}[x, y]$ which is a variable of $\overline{\mathbf{k}}[x, y]$, i.e. which admits $g \in \overline{\mathbf{k}}[x, y]$ such that $\overline{\mathbf{k}}[f, g] = \overline{\mathbf{k}}[x, y]$. Let $G := \text{Gal}(\overline{\mathbf{k}}/\mathbf{k})$ be the Galois group of the extension $\overline{\mathbf{k}}/\mathbf{k}$, and assume that the \mathbb{G}_a -action is non-trivial. Then $\overline{\mathbf{k}}[f]$ is the subring of \mathbb{G}_a -invariant polynomials of $\overline{\mathbf{k}}[x, y]$ and since the \mathbb{G}_a -action is defined over \mathbf{k} , the subspace $\overline{\mathbf{k}}[f]$ is invariant under G . Thus, the action of G on $\overline{\mathbf{k}}[f, g] = \overline{\mathbf{k}}[x, y]$ is given by

$$\sigma(f) = a_\sigma f + c_\sigma, \quad \sigma(g) = b_\sigma g + d_\sigma$$

where $a_\sigma, b_\sigma \in \overline{\mathbf{k}}^*$, $c_\sigma \in \overline{\mathbf{k}}$ and $d_\sigma \in \overline{\mathbf{k}}[f]$. It is enough to show that the 1-cocycle

$$G \rightarrow J_n(\overline{\mathbf{k}}) \simeq (\overline{\mathbf{k}}^*)^2 \ltimes (\overline{\mathbf{k}} \ltimes \overline{\mathbf{k}}[f]), \quad \sigma \mapsto (a_\sigma, b_\sigma, c_\sigma, d_\sigma)$$

is a 1-coboundary. The vanishing of $H^1(G, J_n(\overline{\mathbf{k}}))$ follows from the vanishing of $H^1(G, \overline{\mathbf{k}}^*)$ (see [NSW00, (6.2.1) Theorem]) and from the vanishing of $H^1(G, \overline{\mathbf{k}}[f]) = \varinjlim_n H^1(G, \overline{\mathbf{k}}[f]_{\leq n})$ (see [NSW00, (1.5.1) Proposition] and [NSW00, (6.1.1) Theorem]) by using exact sequences (here $\overline{\mathbf{k}}[f]_{\leq n}$ denotes the polynomials in f of degree $\leq n$). \square

Lemma 4.7. *Assume that \mathbf{k} is perfect. Let $\Gamma \subseteq \mathbb{A}^2$ be a closed geometrically irreducible curve that is defined over \mathbf{k} and assume that it is preserved under a non-trivial \mathbb{G}_a -action (defined over \mathbf{k}). Then there exists an automorphism $\varphi: \mathbb{A}^2 \rightarrow \mathbb{A}^2$ such that $\varphi(\Gamma)$ is an affine line in \mathbb{A}^2 .*

Proof. By Proposition 4.6 (in case $\text{char}(\mathbf{k}) = p > 0$) and by [Ren68] (in case $\text{char}(\mathbf{k}) = 0$) we can conjugate the action to an action of the form $(t, x, y) \mapsto (x, y + p(t, x))$ where $p \in \mathbf{k}[t, x]$ is a non-zero polynomial. Hence, every geometrically irreducible invariant curve is a line in \mathbb{A}^2 . \square

Lemma 4.7 implies that the second case of Lemma 4.3 does not occur. The proof of Theorem 2 is now clear:

Proof of Theorem 2. By Theorem 1, either Γ is a line or $\text{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic subgroup of $\text{Aff}(\mathbb{A}^2)$ or J_n for some $n \geq 1$. If $\text{Aut}(\mathbb{A}^2, \Gamma)$ is an algebraic group of positive dimension, it contains a closed one-dimensional torus by Lemmas 4.3 and 4.7. The description of the possible cases follows then from Proposition 4.5. \square

4.4. The case of finite groups. There are plenty of examples where $\text{Aut}(\mathbb{A}^2, \Gamma)$ is finite. The simplest way to get such examples is to take a finite subgroup $G \subset \text{Aut}(\mathbb{A}^2)$ and to look for invariant curves. Since G has a finite action on $\mathbf{k}[x, y]$, one can find a lot of invariant polynomials, and in practice most of them are irreducible.

In characteristic zero, the group G is reductive and thus contained, up to conjugation, in GL_2 (see [Kam79]). In positive characteristic, there are however plenty of non-linearisable subgroups of $\text{Aut}(\mathbb{A}^2)$, and, as far as we know, there is for the moment no classification of the conjugacy classes of such subgroups.

Example 4.8. Let \mathbf{k} be of characteristic $p > 0$. For any integer $a > 1$

$$\varphi: (x, y) \mapsto (x, y + x^a)$$

is a non-linearisable automorphism of order p of \mathbb{A}^2 , which preserves the family of curves of the form

$$y^p - yx^{a(p-1)} = q(x),$$

where $q \in \mathbf{k}[x]$.

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MATHEMATISCHES INSTITUT, UNIVERSITÄT BASEL, RHEINSPRUNG 21, CH-4051 BASEL
E-mail address: `Jeremy.Blanc@unibas.ch`

MATHEMATISCHES INSTITUT, UNIVERSITÄT BASEL, RHEINSPRUNG 21, CH-4051 BASEL
E-mail address: `Immanuel.Stampfli@unibas.ch`